DIVISIBLE AD INFINITUM

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I

No philosophical problem has evoked as widespread and keen an interest as have the arguments of Zeno, son of Teleutagoras, the Eleatic philosopher of fifth century B.C.[86] Surprisingly, however, little interest has been evinced by philosophers or mathematicians in the hypotheses of atomicity and/or infinite divisibility—one or the other of which has been accepted by every interpreter as the hypothesis on which the arguments proceed or as the one on which the arguments would be valid—even though each one of these hypotheses is of immense interest in itself and one or the other of which has to be postulated with regard to the constitution of Space/Time in particular and of a pluralistic unite in general.[87] The present writer does not recall any comprehensive attempt at the analysis of either of these hypotheses; in fact, we know of only two detailed discussions of the hypotheses in question, the one being the Peripatetic critique of atomicity,

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[86] In our book, Towards A Definitive Solution of Zeno's Paradoxes (Karachi, 1973), we have listed nearly 200 works on Zeno's Paradoxes which is by no means an exhaustive list of even the works published in English !

[87] . By "a pluralistic unit" we shall mean an x such that x is a unit in its own right (with all the implications that the word "unit" carries with it) and is yet a "whole" consisting of distinguishable "parts," or such a "collection" of (discrete) units as is nevertheless a unit in its own right.

De Lineis Insecabilibus,[88] and the other being Hume's critique of infinite divisibility contained in his Treatise.[89]

With regard to the constitution of a pluralistic unit, apart from the atomistic hypothesis proper (the hypothesis that any pluralistic unit is composed of a finite number of indivisible units each of unit magnitude, hereinafter to be referred to as the "finposatomic hypothesis") and the hypothesis of infinite divisibility (the hypothesis that for any value of x, if x is a pluralistic unit or a part thereof, then x is divisible, hereinafter to be referred to as the "infible hypothesis"), two other hypotheses have been

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[88] Peel, Atomon Ghrarnmon (On Indivisible Lines) is justly regarded as a work by some writer of the Peripatetic School, but not to be a work of Aristotle himself. The treatise contains arguments against the possibility of there being "indivisible lines" (finposatomic hypothesis) as well as against the assumption that lines are constituted of "points" and time of "vows"—but it is not made clear whether a finitude or an infinitude of them supposedly constitutes the lines. (The author does not indicate what view he was defending. Obviously, he took it for granted that there could be only three possible views, and since two of them were assumed to get demolished by his arguments, the view expoused by the author could be taken as having been established. It is also obvious that that third view must have been an unconsciously held version of the imposinfible hypothesis.) Most of the arguments advanced are invalid as against the finposatomic hypothesis, though some of the arguments are of more than merely historical interest.

[89] David Hume (1711-1776), A Treatise of Human Nature (London, 1733, reprint, London, 1961), pp. 34-59. Of his numerous arguments against the hypothesis of infinite divisibility only one presents a serious difficulty in assuming space/time to be infinitely divisible, viz. the difficulty that no period of time would uniquely qualify to constitute the present. (We shall present this difficulty in our own way in Sections III and IV. However, another of his arguments presents a genuine shortcoming of the imposinfible hypothesis—which is what Hume naturally took to be the hypothesis of infibility, viz. that of imprecision and lack of absoluteness in the notions of "equality" and "inequality," which, in conjunction with other shortcomings, shows the imposinfible hypothesis to be inadequate for purposes of Science and Mathematics. Although not one of his arguments against the infible hypothesis, in arguing against the notion of (what Hume calls) "mathematical points," Hume has presented what is in fact the only serious difficulty in assuming anything, space, time, or the universe, to be posinfible, viz. the difficulty that the "points," "moments" and the like, are "plain nothing" (are magnitudeless and do not form any parts of any positive intervals) and yet must in some sense be contained in positive intervals. Hume's other arguments, it is contended, are inconclusive when not fallacious (Hume's main fault lies in his failure to distinguish between an idea in the sense of a concept and an idea in the sense of an image. In addition, he assumes that space/time being real must behave like material bodies. More-over. he unconsciously assumes that "the parts of x" is a meaningful expression even where x is supposedly infible.)

advanced which are obtained by a modification of the finposatomic and/or the infible hypothesis: (1) the hypothesis of there being parts of infinitesimal magnitudes[90] of which any pluralistic unit is constituted, postulated by the seventeenth-century founders of the Infinitesimal Calculus, and (2) the hypothesis of there being parts of null magnitudes of whose super-denumerable infinity[91] any pluralistic unit is constituted, hereinafter to be designated as the "infinzeratomic hypothesis," postulated by the nineteenth-century founders of modern mathematics. The hypo-thesis of there being infinitesimal parts of which any unit is constituted, apart from its having been discarded by mathematicians in favour of the infinzeratomic hypohthesis, is founded on the fallacy of appearing to define an actual entity while in fact only defining a hypothetical relation between an actual and a hypothetical entity. If y is of positive magnitude, then there is no x such that x is infinitesimal with respect to y, though we know what it would be like for x to be infinitesimal with respect to y if there were such an x. We have, therefore, not considered it worth our while to discuss this discarded hypothesis. We have discussed infinzeratomic hypothesis at considerable length in an earlier work, reaching the conclusion that the hypothesis involves a self contradiction.[92] The finposatomic hypothesis, we have shown,[93] involves no

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90 Any number or any measurable quantity or magnitude A is said to be infinitesimal with respect to any other number or any other measurable quantity or magnitude B, if A is greater than zero and if for any value of n (n=1, 2, 3, . . .)n times A is less than B. Loosely speaking, for any value of y, if y is of positive magnitude and y>x>O, then x is of infinitesimal magnitude.

91 Sets whose cardinality is greater than that of the set natural numbers are said to be "super-denumerable" or "non-denumerable" sets. One would not suppose that there could be a set whose cardinality was greater than the cardinality of the set of natural numbers or that one infinite set was numerically different from another infinite set. But Georg Cantor has offered a "proof" to establish that the cardinality of the set of real numbers is greater than that of the set of natural numbers. Many mathematicians did not accept the proof. I believe, those who accept Cantor's postulates are in the wrong in not accepting his proof ; but, I maintain. that the proof is vitiated by t the assumptions that (I) any given set can be the set of natural or real numbers, and that (2) if there is a one-to-one correspondence between two series A and B, we can say, even when A and B are infinite, that there are as many elements of A as of B, and that (3) an infinite sequence turned set has any cardinality at all. (The assumptions are the same as those of infinzeratomicity.)

92 See "lnfinzeratomicity," The Pakistan Philosophical Journal, Vol. X[II, No. 3 (October 1975), pp. 47-84, and Vol. XIII, No, 4 (December 1975), pp. 34.72.

self-contradiction or any other insurmountable (logical) difficulty—all the arguments heretofore urged against it being demonstrably invalid—but the hypothesis is not satisfactory enough for purposes of Science and Mathematics. In what follows, we propose to discuss in detail the remaining alternative, the infible hypothesis.

We shall divide our discussion into two main parts. In the first of these we shall present a number of difficulties encountered in conceiving, or maintaining, something to be an infible unit or a part of a supposedly infible unit. Some of these difficulties have been stated by earlier writers; we are including them here partly to make our discussion comprehensive and partly to be able to show how best these difficulties can, in our opinion, be surmounted. The other difficulties we shall venture to present as arguments against the adoption of the infible hypothesis as such or in one of its more specific forms, the hypothesis of imposinfibility and posinfibility.[94] [By "imposinfible hypothesis" we mean the hypothesis that no such mngnitudeless things as "points" or "moments" are in any manner contained in any (supposedly) infible unit (of positive magnitude). In fact, on this hypothesis, there can be no (geometrical) "lines" or even "surfaces" (though there may be one-dimensional continua of other types), for, otherwise, there would be "points" too: two lines intersect in a point, and two surfaces intersect in a line. By "posinfible hypothesis" we mean the hypothesis that such magnitudeless things as "points" and "moments" are contained in (such supposedly) infible units (as "lines" and "periods" of time), their infibility notwithstanding, not as parts of units but as limits of and joints between any two parts of spatio-temporal intervals. A surface, on this hypothesis, is the limit of and joint between two solids, a line is the limit of and joint between two surfaces (i. e. two surfaces intersect in a line), and a point is the limit of and joint between two segments of a line (i.e. two lines intersect or meet in a point, and two contiguous segments of a line share a common point)]. We shall preface these difficulties—which presuppose there being no self-contradiction involved in the notion of a "pluralistic unit" with the more

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93 See "The Atomistic Hypothesis Reconsidered," The Pakistan Philosophical Journal, Vol. XIII, No. 2 (Jan. June 1975), pp. 14.42.

94 All these difficulties have appeared to me, at one time or another, to be compelling reasons for the rejection of infibility in general, or in one of its two specific forms of imposinfibility and posinfibility.

general difficulty of conceiving as a pluralistic unit This difficulty was encountered as soon as the Greeks started philosophising, but to which, in its purely logical form, it is con-tended, no definitive solution has yet been offerred. In the second part, we shall endeavour to show how all these difficulties can be overcome, and shall go on to argue that the infible hypothesis, in its specific form of posinfibility, is the one which is not only presupposed both in (Euclidean) Geometry and the natural languages but which is, for purposes of Science and Mathematics, also the most satisfactory of all the hypotheses regarding the constitution of a pluralistic unit or, what is the same, regarding "whole"-"part" relationship,

II

If the whole Universe, Time or Space-Time, etc., is not conceived a la Parmenides as a simple unity devoid of all multiplicity—as the Parmenidean One—but as something capable of accommodating the being of the "Many,"[95] then the relationship

between the One and the Many with ragard to origination must be conceived of in one of two ways: (i) the One is given, and the Many arise there from by the process of division (e g. the Universe is given, the individual things arise as a result of division, actual or conceptual), and (ii) the Many are .the ones that are given, the One arising there from as a result of their aggregation (e.g. the individual things are there, the "Universe" is the actual or conceptual aggregation of these things). But whichever of these two views we take, we find a great difficulty in conceiving of the relationship between the One (whether given as a unit in its own right or supposedly obtainable from things given as units) and the Many (whether given as units in their own right or supposedly obtainable from something given as a unit) —that is to say, we find it difficult to have both the One and the Many, no matter with which of these two do we begin.

If we begin with something given as a unit and Endeavour to splite it up into a plurality of parts, then we are presented with a very difficult problem. If we imagine that the unit has actually been broken into a number of parts, then the unit ceases to be a unit properly so called and is transformed into

[95] It is not really an ontological question. The problem is to let a unit, with all its implications, have parts, or to let a number of units give rise to a whole which can be a unit in its own right.

being a "collection" of discrete units, and, obviously, its unity and continuity disappear. If, however, we imagine the One as transformable into the Many not actually but conceptually, then the unity and continuity of the unit is retained but the unitness and discreteness of the parts obviously become questionable. In other words, if really discrete units are obtainable from what has been assumed to be a unit, then it is obvious that what we really have as given is a collection of units (a whole) and not a unit; if discrete units are not obtainable, then the One remains, undisturbed in its unity and continuity, but the Many fail to arise therefrom.[96]

If, on the contrary, we begin with a number of units and endeavour to obtain such a whole from them that it can be taken as a unit in its own right, then again it seems to be a hopeless task. All that we seem able to achieve is a collection of units so placed or disposed that there is an appearance of unity and continuity but where there is no real unity. The units do not "gnaw" into each other or interpenetrate each other. They remain united, like the beads of the rosary, only for so long as a string runs through them (or for so long as we keep them together in our thought) and gives them a semblance of unity, which disappears as soon as the string is removed. Some sort of string is necessary to unite the discrete units. But what can serve as the string, the link or the bond? It cannot be just another unit, say, B, between A and C—for, if it were just another unit, then again there would be needed a unit between A and B, and another unit between A and C.

Thus, if we begin with a given unit, we fail to obtain such parts thereof as would be units in their own right without destroying the unitness of the given unit; if, however, we begin with a number of units as given, then we fail to so reassemble them as to give rise to an aggregation which could be a unit in its own right.

III

[96] This I believe to have been at the back of the controversy between the Greek Monists and Pluralists, even though their discussion is suffused with the ontological idiom, which hides the fact that what is at stake is the conceiving of an "indivisible whole" (a question of logical analysis) and not that of there being or not being in reality more things than one.

If there is to be an intelligible discourse, not to mention Science and Mathematics, the conceptual difficulties presented above must be resolvable, though we may not be able to see how the difficulties are to be resolved.[97]

If we do assume that the difficulties are resolvable, then we presuppose either that there can be an x such that x is a unit in its own right and is capable of being resolved into a set of parts each one of which is itself a unit in its own right, or that there can be a set of things each one of which is a unit in its own right and yet their aggregation can give rise to a whole which is a unit in its own right. In other words, if we do not enter into the question of primacy,[98] then we may say that we assume that there can be a "whole" (which is a unit in its own right) constituted of a number of "parts" (each one of which is a unit in its own right). Now the whole (given by itself, or given as constituted of a number of units) may be assumed' to be such that it can be resolved into indivisible components, or to be such that it cannot be resolved into indivisible components, i.e. can be resolved into only (further) divisible components. The former assumption we

have referred to as "finposatomic"[99] and the latter as "infible", As stated before, we have discussed the finposatomic hypothesis elsewhere. Here, we propose to present the difficulties one would encounter in assuming something to be an infible unit.

The problem of obtaining "parts" from a given "unit" (or a "whole" from a number of given "units"), as we just saw, is in itself very serious. But this problem gains in seriousness if we endeavour to obtain such parts from a

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97 No earlier writer known to me (has) really succeeded in resolving it, because, I believe, the problem was not consciously seen to be a purely logical matter and because it was 'not realised that the solution of this problem was dependent upon the postulation of either the infible or the non-infible (atomistic) hypothesis (and was variable depending upon whether the former or the latter was the hypothesis postulated).

98 The question being: with which can we begin, a unit or a set of units?

99 We are here concerned with component parts and not with what we may refer to as constituent parts. Though we could make the assumption that there are indivisible parts (components or constituents), to which we should have referred as the atomistic assumption, I find it fruitless to do so, since, I believe, we have conclusively shown that the infinzeratomic hypothesis (the hypothesis that a positive interval is constituted of an infinitude of indivisible constituents) involves a self-contradiction. See "Infinzeratomicity," op. cit., Vol. XIII. No. 3 (October 1975), pp. 47-84, and Vol. XIII, No. 4 (December 1975), pp. 34-72.

given unit that the parts are not only units in their own right but are also further divisible into ever divisible parts (or, if we endeavour to obtain a whole from a number of units each one of which is itself composite)[100]

(1) To begin with, we would be faced with the difficulty that the parts, how small so ever their magnitudes, cannot be regarded as real units because of their being divisible themselves. If we resolve a given unit x into a set of parts, xi, x2, x3 ... xn, then, on the infible hypothesis, none of x1, x 2 n can be regarded as a unit properly so called, for each one of them is resolvable into sets of parts and, as such, is a whole and not a unit. Whatever be the magnitude of the unit, and whatever be the magnitude of xi, x2, etc., these are, ex hypothesi, divisible, and, hence it would seem that they cannot be accepted as units in their own right.

(2) If x is of finite magnitude, and it is assumed that it is divisible ad infinitum, then one of two cases must be assumed: that (a) the process of division can get completed, or that (b) the process of division cannot get completed. But neither (a) nor (b) can be upheld, and hence it is impossible to accept the infible hypothesis.[101]

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[100] The problem of having such a whole and such parts that the "whole" is a unit in its own right and each of the "parts" is also a unit in its own right, can more easily be surmounted by adopting the finposatomic hypo-thesis. The unfitness of the ultimate "parts" is then unquestionable; the unitness of the "whole" is seen, in so far as space, time, and other mathematical abstractions are concerned, to reside in its continuity, and continuity is found on the finposatomic hypothesis to he nothing other than gapless continuity. The whole on that hypothesis, from the infible point of view, is but a logical fiction—but the whole is not required on the finposatomic hypothesis to be anything else. Whatever is stated about a given whole can be translated in terms of the constituent units.

[101] G.E. L. Owen, "Zeno and the Mathematicians," Proceedings of the Aristotelian Society, N.S., Vol. LVIII (1957-58), pp. 199-222, has rightly presented Zeno's "Metrical argument" in conjunction with his "Dichotomy argument". As will appear in the course of our paper, Zeno's argument could certainly have been of this form :

(1) If x is infible, then the process of division is either :

(a) capable of being exhaustively carried through, or,

(b) is not capable of being terminated.

(2) If (a), then (i) the parts would either be magnitudinous or (ii) magnitudeless ; but if (i), then the whole must be infinite in magnitude, and if (ii), then the parts are plain nothing, do not exist at all, and cannot give rise to a whole of positive magnitude. If (b), then it would be impossible to traverse any distance, for, to be able to traverse any distance, it is necessary to traverse an infinitude of distances. That it is necessary to traverse an infinitude of distances can be seen by considering a race between a faster and a slower runner.

(al) That the process of division can get completed, cannot obviously be maintained without self-contradiction: is it not self-contradictory to maintain that an endless process comes to an end?[102] How can an endless process come to an end, since one can never get any nearer to completing the process?[103] Obviously, an infinite process cannot get completed.

(a2) Moreover, it is clear that the infinite process cannot be completed without an infringement of the generating principle. Sometimes commercial organizations employ persons on condition that they will receive a certain salary and the governmental tax thereon shall be paid for by the company. In company A, Mr. B was employed with the stipulation that Mr. B will receive a tax-free salary of Rs. x per annum. Fortunately for the accountant of the company, income-tax for the year was on the flat-rate basis of per rupee. The accountant started calculating the gross salary to be given to Mr. B, so that, after paying the income-tax, he could get the stipulated salary of Rs x. The accountant first wrote down "Rs x" and then added ((x)} rupees—the amount of tax on Rs x. The accountant then realised that the income-tax officials would not be satisfied with {_-_ b (x)} rupees, for the gross salary having increased to {(x)-#- b (x)} rupees, the tax due on the gross a

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[102] Mathematicians would maintain that it is self-contradictory only in the etymological sense of "endless" or 'infinite," not in the sense in which a given (endless) process has a determinate number (a transfinite number) of stages. All the stages of the "endless" process, they maintain, come to an end when the relevant period of time comes to an end. [See, e.g., R.M. Blake, "The Paradox of Temporal Process," The Journal of Philosophy, Vol. XXIII (1926), pp. 645-52, and Bertrand Russell, "The Limits of Empiricism," Proceedings of the Arisroteilan Society, N.S., Vol. XXXVI (1935-36), p. 131.] We have argued against mathematicians' view in "Infinzeratomicity" (op. cit.) and regard those arguments of ours as conclusive.

[103] If x be any given stage, there would be just as many stages of the remaining process as there were at the very outset, for, if x is a given stage, then there would be only a finite number of stages between x and the first stage. Hence, at no given stage would anyone be any nearer to completing an endless process than he would be at any other stage, no matter how many stages there be between those two stages. Chas there is a legerdemain suddenness in the completion of an endless process. [Supporters of mathematicians' views too have felt this suddenness ; see, e.g., J. Watling, "The Sum of An 1nfinite Series," Analysis, Vol. XIll (1952-53), p. 46.] We have argued that this suddenness in completing an infinite process comes from the fact that what gets completed is a finite process such as traversing a finite distance which is by assumption turned Into the completion of an endless process such as traversing an infinituue of (component) distances ; that, in fact, no "endless" processes at all come to an end. (ace "infinzeratomicity," op. cit, Pt. 1, pp. 78-84.)

a2 salary amounted to € (x)+ (x) rupees, or, b (x)}a b b2 rupees plus {_- bz (x) rupees. So he had to add up the latter amount also. But again, some tax had to be paid on the last mentioned amount. So he incorporated that amount too in the gross salary. But then some tax had also to be paid on the last addition to the gross salary, ... It would seem that the accountant cannot determine either the gross salary or the tax thereon without simultaneously determining the gross salary and the tax thereon. We may present here a section of the entries made by the accountant.

Net salary Income-tax to be payable to paid by the coin- Cross salary B piny on Mr B's y payable to B (in rupees) gross salary Mr (in rupees) _en 1. x a i - — 2. (x) I 2 3. x+_b (x) 4. b2 (x) 5. x-{ a (x) + a2 (x) 6. ...7. z Total x --a- (x)+ b2 (x) +...3 x-{- - (+ 22 (x)+..._ea

It is abovious that, irrespective of the length of the register and the time at the accountant's disposal, he will not succeed in bringing his calculations to a satisfactory end.[104]

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104 Mathematicians would, however, claim that the accountant can bring his calculation to a satisfactory end. When an Infinite number of entries have been made in either Column 2 or 3, the whole operation would come to a successful end—both the gross salary and the tax payable thereon shall have been calculated. The only difficulty they find here is that of the reflexiveness property, the whole being no greater than the (proper) part.

When all the entries in Column 2 or 3 have been made, and as such no entries remain to be made, it is obvious that the gross salary as also the tax payable thereon shall have been determined. Therefore, the only question is whether all the entries can be made, and it seems obvious that it cannot be done without an infringement of the generating principle Mathematicians, however, make an affirmative answer possible ; and this they do by a simple device—the times taken to make later entries become progressively shorter in the form of a 2-sequence. Thus, at the end of a finite period of time, a whole infinity of calculations gets finished. Some philosophers thereupon came up with examples of infinite series which had no natural (i.e. logical) limit : the infinity machines and the Hercules-Hydra Ordeal (M. Black, Analysis, Vol. XI (1950-51), pp. 91-101, and the series of on-off switchings of a lamp (J.F. Thomson, "Tasks and Super-Tasks," Analysis, Vol. XV (1954-55), pp. 1-13].

This obliged mathematicians to distinguish between two types of infinite series, and to maintain that both the types of infinite series can be completed and that in this respect there was no difference between them, the difference lay only in the manner of determining the result achieved : while in one type of series (the Z-sequence type of series) the result achieved (the state of affairs at the w+1th stage) was determinable a priori, in the other type of series (the Hercules-Hydra type) the result achieved had to be determined by convention. Paul Benacerraf tin "Tasks, Super-Tasks and Modern Eleatics," The Journal of philosophy, (a3) If, however, the process is assumed to get completed, then the question arises as to the result thereof—whether or not a set of parts results?[105] (a3a.) It would defy our imagination that no set of parts results. (What happened to x, or to the parts or sub-parts into which it was resolved at one stage? They cannot just vanish. Each part is divided into shorter parts such that the set of parts is equal to x.) (a3b.) If a set of parts results, then the question is whether the resulting parts are finite or infinite in number. It is obvious that x must in principle be capable of being divided into art infinitude of parts. Since every (positive) part of x is, ex hyphothesi, divisible ad infinitum, it would seem that the number of the parts of any infible unit (obtained by exhaustive division) cannot possibly be finite; for, otherwise, only a finite number of divisions would be required to reach the parts, and, as such, only a finite number of divisions would, contra hypothesis, be possible. (If n be the number of parts, then the number of divisions required

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Vol. LIX (1952), pp. 765-84), argued that the result achieved, if any, was altogether irrelevant ; there might not in fact be any result at all—for example, the genie engaged to write out the series of natural numbers may get reduced by half at every next stage (being of full height at the start, reduced to half after/in writing "one," reduced to a quarter of his original height after/in writing "two" and so on). We have strengthened this line of argument by maintaining that on the imposinfible hypothesis, the result achieved was simply the completion of the relevant infinite process, and that on the posinfible hypothesis the result was the attainment of the goal, "the limit," but here "reaching the goal" added nothing substantial to the imposinfible result of the completion of the infinite process, and, finally, that it was because of the assumption of infinzeratomicity that a problem arises, which, however, is overcome by the fact that the w±1th term adds nothing to the w-sequence of terms. In fact, the problem of "what is the result achieved" does not arise where a really infinite extension is involved : the w+1th term need not be postulated at all ; where the infinity involved comes from infibility, there will have to be a w+1th term, but there a convention regarding the w±1th term would help resolve the problem.
We have, however, gone on to argue that the infinzeratomic hypothesis is self-contradictory. Hence, the endeavour to meet the Case of the Obstinate Accountant is seen to be a failure. and the difficulty presented by the Case of the Obstinate Accountant gets rehabilitated.

[105] This is different from asking the question as to what results. We are here asking whether any parts at all result ; the other question is : "What parts result?" i.e. "of what magnitude are the parts that do result ?" That the process of division must yield a set of parts is obvious, but of what magnitude must the parts be when the process of division has been exhaustively carried through has no answer, for the process of division, on the infible hypothesis, cannot be exhaustively carried through. (If, however, it is assumed that the process has been exhausted, that the resultant parts must be a continuum of zero-magnitudes. If one limit is, ex hypothesi, attained, the corresponding limit must also be attained.)

would be only n-1.) Hence, the number of the parts of x cannot possibly be finite. Therefore, we must assume that the number of the parts of x is infinite.[106] Now, the question is whether the (infinitude of) parts into which x is (in principle) resolvable are to be assumed to be of positive magnitudes, however small, or they are to be regarded as of no magnitude, It would seem that neither offers a tenable alternative. (a3bi.) We cannot assume that the parts are magnitudinous, for, (a3bia,) it would be in-compatible with our assumption that the process of division was completed—the parts of a completed process of division must be magnitudeless, for if y has any magnitude, then it would be, ex hypothesi, (further) divisible.[107] If y be one of the parts in question, then it is either magnitudinous and, as such, must be further divisible (and hence the process of division is shown not to have been exhaustive), or it is not further divisible and, as such, must be magnitudeless[108] (a3bib). Moreover, if the (infinitude of the) parts be

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106 D. Hume and G.E.L. Owen both base the conclusion of infinitude of parts on the premises of division being otherwise finite in number. [In "Zeno's Paradoxes. Towards a Solution at Last," Islamic Studies, Vol. XI (1972), pp. 125-51, however, we have argued that the number of the parts of x must be infinite if it is assumed that there must be some number of the parts of x, or even that there can be any such thing as "the parts of x". (Hume seems to have also assumed that there must be some number of the parts of x.)]

107 if y is a magnitudinous component of supposedly infible unit x, then y must be divisible, for, otherwise, x wound not be infible.

108 This may be referred to as the 'Either-Divisible-or-Magnitudeless" argument. The argument is valid on the infible hypothesis, for, on this hypo-thesis, "to he magnitudinous" implies "to be divisible," and "to be divisible," on any hypothesis, implies "to be magnitudinous".

One of Zeno's arguments reported by Simplicius 'in Physics, 138, 18-19) can be so construed as to proceed on the "Either-Divisible-or-Magnitudeless" argument and lead to the claim that a supposedly infible unit must either be infinite in magnitude or be of no magnitude at all. The parts into which x has been resolved, xi, x2, x3, ... are either divisible or are magnitudeless; if x1. x2, x3, . . . are magnitudeless, then they are plain nothing and do not exist at all, and hence x must itself be magnitudeless and as such non-existent ; if, however, xl, x2, x3, . . . are divisible, then they must be magnitudinous, and hence their aggregate must be infinite in extent (assuming that "al+a2+a3+ . . ." must give rise to an infinite magnitude, if al. a2, a3, ... be magnitudinous). I did in fact so construe Zeno's argument in "The Atomistic Hypothesis Reconsidered" (see. p. 30) ; but I now think Zeno's argument to have been simply that if x is infible. there must be an infinitude of parts, which (i.e the parts) if supposed to be magnitudeless would not exist at all, but if supposed to he magnitudinous would give rise to a whole of infinite magnitude [l was probably misled by the fact that each of the constituent units is said by Zeno to be magnitudinous ; I thought that there being a magnitudinous, then the addition of the (like) magnitudes of an infinity of parts must, contra hypothesis (that x is of finite magnitude), give rise to an infinite magnitude[109] (a3biia). But we cannot assume the parts to be magnitudeless either. For (a3biia) it would be incompatible with the assumption of infibility---that, for any value of y, if y is a part of x, then y is divisible—and hence y must be magnitudinous.[110] Moreover, (a3biib) if y is magnitudeless, then y is plain nothing and therefore y cannot exist at all.[111] Furthermore, (a3biiC) if the infinitude of parts into which x has been supposedly resolved are all magnitudeless, then the addition of their (zero) magnitudes cannot 'yield a whole of positive magnitude, for "0+0+0+ . .." is equal to "O" and x would have been, contra hypothesis, proved to be of no magnitude.[112]

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successor to a given constituent unit was a consequence of its being magnitudinous, assuming that the successor unit was a constituent sub-unit of the given constituent unit. Zeno, on the contrary, seems to have argued that. just as in the progressive interpretation of the Dichotomy, there will be an infinitude of succeeding constituent units and that each of those units must be magnitudinous (for otherwise they would not exist at all!), whence Zeno arrived at the conclusion that the supposedly infible unit must be infinite in extent.]

109 This would seem to have been one part of the argument referred to by Simplicius. This, however, has been challenged by mathematicians (see, infra, Note 27.

110 The argument may prima facie be met by distinguishing between parts which are components and parts which are' constituents. But. as we have shown in "Infinzeratomicity" (op. cit.) such a distinction would in the end be of no avail.

111 This is one part of Zeno's argument reported by Simplicius. Mathematicians who do postulate such ys do not seem to me to have met the argument ; if pressed, however, they would probably retort that "But, ys do exist," and offer the continuum of real numbers as an example of a surer-denumerable infinity of degenerate intervals each one of which can he singled out, and hence cannot be. said not to exist at all.

112 This follows from Zeno's argument, even if the purport of the argument be not to this effect. This argument, however, has been challenged in recent times. Modern mathematicians do not regard the argument as valid, maintaining that an infinite set of intervals each of whose members is of finite/positive magnitude may give rise to an interval of only a finite magnitude (such as the set of the distances of a Z-sequence) and that a non-denumerable set of degenerate intervals may give rise to an interval of positive magnitude [see. e.g,, A. Grunbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements," Philosophy of Science, Vol. XIX (1952), pp. 288-306].

The parts, if members of a Z-sequence, would be characterised by the peculiarity that none of them is the smallest, and their aggregation would, therefore, not be like an aggregation in which each part is in magnitude equal to or greater than a given magnitude. The aggregation of such parts may or may not be equal to the given finite interval—to be more precise, the sum of the members of the aggregation may give rise to a magnitude

Thus, we cannot assume the number of the parts to be either finite or infinite, nor can we assume the resultant parts to be either magnitudinous or magnitudeless, and, hence, we cannot assume that a set of parts results as a consequence of the process of division. But either a set of parts results or a set of parts does not result from the exhaustive process of division, and,

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which is equal to or less than (but not greater than) the given unit—they cannot give rise to an infinite magnitude. If x be a member of such an aggregation, then there are/is only a finite number of the aggregation's members who are greater than x, while an infinitude of members is less than x. Even if we take the members out of the sequence and put them into a set, the position remains unchanged : given a member of any magnitude, an infinitude of members are of less magnitude than the given member's magnitude. If, however, we change the method of division—if, instead of dividing only one of the two parts into which any given part is divided, we divide both of the two parts and then each of the four parts and then each of the eight parts, and so on—then it appears at first glance that Zeno would have been vindicated. Not at all. If we have a set of members after the whole operation has somehow come to an end, then we think that since for any value of x, if x is a member of the set, then x has a positive magnitude, the addition of the magnitudes of all the members must yield an infinite magnitude. But we would forget that for any value of x, if x is a given member, then there is an infinitude of members such that each is of less magnitude than x. However, it might be asked as to what happens when such an exhaustive process of division has taken place; do we not have components of positive magnitudes? The answer is, we do not have such components. We reach the magnitudeless constituents. How to build up the given unit from the magnitudeless parts? Zeno is partially right: even an infinite (denumerably transfinite) set of magnitudeless parts ,cannot give rise to the given unit. These parts cannot obviously be added the way a set of given positive magnitudes can be added. But if the magnitudeless parts are laid out, then it is necessary that there should be no holes, which can be assured only if the degenerate parts be non-denumerably transfinite, or, in other words, only when all the degenerate parts between any two given parts are laid out. (Grunbaum is right in maintaining against Russell that what is philosophically important is not the compactness of points but the super denumerability of points.)
1t would thus seem that both the conclusions that the sum of the magnitudes of the infinitude of parts must be infinite, and that the sum of the infinitude of magnitudeless constituents must be equal to zero) are nonsequitor. If we do assume that an infinite set of components results, a view against which we have argued in "Infinzeratomicity," then mathematicians are to this extent right that the sum of the components cannot exceed that of the given interval. Again, if we do assume that a finite interval is constituted of an infinitude (super-denumerable infinity) of degenerate intervals. which too we have argued against in "Infinzeratomicity," then mathematicians would be right in maintaining that the "sum" of the magnitudeless elements can give rise to a finite magnitude. But, as stated in the paper referred to above, their views are based on unacceptable assumptions. Hence, the arguments presented in the text get rehabilitated.

since neither alternative is tenable, the assumption 'that the infinite process of divisions gets completed must be given up.

(b) If it is assumed that the process of division cannot be got completed—and this is what would seem to be entailed by the assumption of infibility[113]—then a set of three considerations would seem to make the acceptance of infibility impossible. We shall designate these considerations as the (b) Which-First? (b2) No-Last and (b3) Which-Now? arguments.

(b1) All was set for the Olympic race, and Achilles was tipped to be the winner by a clear margin. One of the competitors, whose name we are not allowed to disclose, engaged a famous dialectician, Zeno, son of Teleutagoras, to get Achilles disqualified from the competition. Zeno called on Achilles and asked him as to what he intended doing the next morning. Achilles told him that he had to run a race. Zeno asked him what Achilles pro-posed to do about that. Achilles told him there was nothing to do about that: the competitors could not muster half as great a speed as he was capable of, and, if he so desired, he could even give them a handicap and win the race. Zeno clarified his question—what distance did Achilles have to run, and with what speed he proposed to traverse the distance? Achilles told him that he had to traverse distance d which he intended to do with s speed. Whereupon Zeno said that Achilles could not do that—he had first to traverse half of d before he could traverse the given distance, d. Achilles agreed, and said that certainly he would first finish the half of d and then run the second half of d But Zeno said, "Before you traverse the first half of d, you must first traverse the first half thereof, that is, the first quarter of d...."

The next morning Achilles appeared with Zeno before the start of the race and asked the Umpire as to which distance he should traverse first of all. Since the Umpire was unable to get the better of Zeno in the ensuing "argumentation," Achilles refused to run unless the Umpire could tell him which distance he should traverse first of all, Achilles, it is obvious, cannot get started, for the track being infinitely divisible or infible, the Umpire is unable to tell him which distance he should traverse first, for, if he proposes any distance however short, that distance would be found to be divisible and

113 That is, "exhaustive division of an infible unit" involves a contradiction in terms; there can be no x such that it is a part of an infible unit and yet it cannot be (further) divided.

a part thereof would have to be proposed first (we shall refer to this argument as the Which-First? Argument/difficulty).

It would seem, therefore, that the hypothesis of infibility must be given up if Achilles is to traverse any distance at all.[114]

(b2) Assuming for the sake of the argument that the above report is apocryphal and that Achilles did run the race, the question is whether he could have succeeded in traversing the given distance, its infibility notwithstanding But to have traversed d is to traverse an infinitude of (part) distances, d1, d2, d3,… And it is obvious that if d cannot be fully divided into d1, d2, d3, . . . without completing the process of division, then d1, d2,

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[114] Zeno's Dichotomy argument interpreted retrogressively. The argument in this form has been quite fashionable in philosophical circles ; see, e.g , Sextus Empiricus. Adversus Mathematicos, 10, 139-41, P.E.B. Jourdain, "The Flying Arrow : An Anachronism," Mind, N.S., Vol. XXV (1916), PP-42-55, A.N. Whitehead, Science and the Modern World (1925, reprinted, New York, 1964), p. 118, and R. M. Blake, "The Paradox of Temporal Process," op. cit., Vol. XXIII (1926), pp. 645-54, esp., pp 646-47.
[In "Zeno's Paradoxes : Towards a Solution At Last," while presenting Zero's Dichotomy argument as the impossibility of enduring the whole of any finite period of time, I wrote : "I wonder why it has not yet occurred to anyone that Zeno long ago provided a very easy way out of mortality !" I must confess that I was then under the wrong impression that it had not occurred to anyone that the Dichotomy argument could be reformulated in terms of enduring a given temporal interval instead of being presented as the problem of traversing a given distance in a finite period of time. I had in mind Aristotle's solution on the basis of one-one correspondence between the parts of space and time, and it occurred to me that such a solution would become pointless if the Dichotomy were to be reformulated as the problem of enduring any period of time. In that case the problem (whether conceived as the traversing of a finite distance by traversing art infinitude of sub-distances or as attaining the fixed goal) will have to be solved on its own, without the ruse of one-one correspondence with a co-variable. (I was convinced that the addition of "infinite time" was an unnecessary interpolation, that the real problem did not stem from there being only a finite time at the runner's/ performer's disposal.) But it has been fully understood, at least since A.N. Whitehead, who clearly stated that "The true difficulty is to understand how the arrow survives the lapse of time" [Process and Reality (New York, 1929), p. 106]. I am ashamed to add that Aristotle himself not only realised that the real difficulty in the apprehension of change on the infible hypothesis was that there was no non-composite unit and that as such there could be no distance which was the initial distance to be traversed [as pointed out by H.R. King, "Aristotle and the Paradoxes of Zeno," The Journal of Philosophy, Vol. XLVI (1949), pp. 657-70 ; King quotes Physics, 237-b 3-6 and refers to 235-b-6 if., etc., in this connection], but also pointed out that passage of time presented the same conceptual difficulty as traversing a distance (see Physics, 263-a 3-b 9).

d3, ... cannot all be traversed either. Hence, if the process of division is not completable, then no distance d can at all be traversed. (We shall refer to this argument as the No-Last difficulty or argument). It would, therefore, again seem that the hypothesis of infibility must be given up if the whole of any distance, however short, is to become traversable.[115]

(63) In relation to time, the difficulty in infibility becomes very acute. To make the difficulty obvious, we might call it the "Which-Now?" problem. While two spatial intervals can co-exist in the sense of being in existence together (at the same time), no two distinguishable/differentiable temporal intervals can co-exist in the sense of being in existence together. No two (non-overlapping) intervals of time, however contiguous, can co-exist! if P1 and P2 be any two non-overlapping temporal intervals and if p1 be the present time, then P2 must lie either in the past or be yet in the future. But, if time were to be assumed to be infible, there would be an in-finitude of non-overlapping temporal intervals co-existing with each other, for, if p be an infible interval of time, then there is an infinitude of sub-intervals, Pt. P2, P3, . . . such that P1. P2, P3 are mutually exclusive parts of p and all are gathered together (in p). Or, to put it differently, if p be the present time—and some period of time will have to be the present time —then an infinitude of period of time, ph p2, P3,… (being parts of p) would be coexistent, even though no two distinguishable periods of time can as a matter of logic co-exist (to constitute the "present").[116]

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115 This is Zeno's Dichotomy argument as usually understood. involving the claim that an infinitude of tasks cannot wholly be performed Some writers have rebutted the argument on the ground that it is only medically impossible, not logically impossible, to perform all of an infinite set of tasks [Sec. e.g, L. Couturat, De l'inftni mathematique (Paris, 1896), p. 462, and Bertrand Russell, "The Limits of Empiricism," Proceedings of the Aristotelian Society, Vol. XXXVI (1935-36), p. 144i] to this rebuttal some have given the rejoinder that there are no infinite sets of tasks of the nature in question, but only a series of finite sets of tasks [e.g. A. Ambrose-Lazerewitz, "Finitism and `The Limits of Empiricism,' " Mind. N.S., Vol. XLVI (1937), pp 382-85] ; we have, however, endeavoured to prove (in "Infinzeratomicity") that there can be no set such that it is a set of the parts of an infible unit or a set of the terms of an infinite series, or the like.

116 This argument, so far as I am aware, is to be found first in David Hume (see D. Hume, op. cit , p. 38) Hume concluded that the "now" must be indivisible, and that time must be supposed to be composed of these (indivisible nows or "moments".

We might be tempted to continue with our arguments against the infible hypothesis and urge for example that on this hypothesis such concepts as "equality" and "inequality" would have no precision or absoluteness. But the validity or invalidity of such arguments would depend upon whether or not we postulate there being points, moments, and the like, in positive intervals (as connecting, or lying between, the components). In other Words, it would depend upon whether we postulate the imposinfible or the posinfible form of the infibility. We shall, therefore, now consider the hypotheses of imposinfibility and posinfibility.

IV

The imposinfible hypothesis, it would appear, is beset with a formidable set of difficulties peculiar to itself.

(1) The foremost difficulty associated with the imposinfible hypothesis is that of inexactitude in such (otherwise precise) concepts as "equality" and "inequality". Let us divide—if not actually, at least conceptually--a given unit into two parts A and B. Now, if A and B are not determinable in terms of indivisible and homogeneous units—and, ex hypothesi, there are no such units on the imposinfible hypothesis here—then how can the one be held to be greater or less than, or equal to, the other? If we have two straight lines (i.e. what ordinarily appears as such to us) placed side by side, how are we to decide whether they be equal or unequal? In some cases, one of the two may be sensibly (visibly or factually) greater—as, for example, when the one is what we ordinarily regard as of one foot, and the other is on the same token of an inch of length--but how shall we decide when the two seem to be equal, or when the one seems to be very slightly bigger than the other? We cannot allow recourse to a smaller unit as measure, for how shall we make sure that the two segments of A and B, respectively, marked out by our measuring unit are exactly equal" For sooth, how shall we make sure that the segment marked out by our measuring unit in either A or B is exactly equal to our measuring unit? There being no smallest part of the lines or parts of A and B, we cannot determine with exactitude and precision their equality or inequality.[117]

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[117] If we were to depend upon cur sense of lapse of time for the determination of and comparison between temporal intervals. then the difficulty would become still greater, i.e the measure of time would be even more imprecise than the measure of space.

(2) Moreover, how would a part be determined and how would two contiguous parts be distiguished? Where would the one part begin and the other come to an end ? Since the unit is, ex hypothesi, imposinfible, we cannot postulate something magnitudeless ("points," "moments" and the like) as "parts" in order to distinguish between the parts of the given unit. If not, then what can distinguish contiguous parts? If we were to postulate another (an intervening) part between the two given parts, then how shall we distinguish between the just postulated part and the other two parts? The problem will remain unsolved, involving us into an infinite regress.[118] If, however, we postulate void between the two parts to be distinguished,[119] then the question is whether the void actually separates the parts or it leaves the parts in contact with each other. If we assume that the void actually separates the parts, then the continuity of unit disappears; if we assume that the parts remain in contact, then what the "void" would be and how would it help distinguish between the parts? Would not the parts remain undiffereatiable in that case, and, as such, be just one whole and not two parts?

Moreover, if there be no magnitudeiess parts to serve as "limits," then the unit itself, it seems, cannot properly be said to be something determinate, for, in that case, we may go on dividing the part that is the last on any set of divisions without ever reaching the last part of the given unit.[120] (If we take a line AB, and divide it into, say, three parts, AC, CD and DB, then neither AC nor DB would be the last part of AB. If we again divide the parts, AC into AE, EF and FC, and DB into DG, GH and HB, then neither AE nor FC, nor DG, nor (FIB would be the last part of AB.)

(3) On the imposinfible hypothesis the problem that any period of time is (further) divisible and, therefore, there can be no period which is uniquely qualified to be "the present" becomes very serious. We can have as short a

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118 Thus, Zeno argued that if things (i.e. parts of (he universe) were many (i.e. if the universe had parts), then there would be an infinitude of them, for, between any two things (i.e parts), there would be an infinitude of things, i e. parts. (See Simplicius, in Physics, 140, 28 D)

119 According to Aristotle (Physics, 2I3-a 12-b 29), natural philosophers postulated the void for two reasons. some because they thought that motion could not take place in a plenum, and some (the mentions the Pythagoreans) to constitute "a kind of separation and division between things next to each other.

120 We are taking it for granted that for a 1inear interval to be determinate it is necessary that it should lie between two parts which serve as the end-parts of the interval.

period as we please, but even this would be divisible and would not qualify to constitute the "now". Thus we can have no knows. But the problem is, if no time is the present time, then nothing can happen in the present; everything shall have happened or will yet have to happen. But surely there must be a present, and something must be happening now, if it is to become something that had happened. All past time is past and gone; the future is yet to be: if there is no present, then what exists, and, again, if there is no present, then what will be the difference between that which (we believe) exists and that which does not? Is not time (as also space) an essential element of existence? Nothing happens in the future; now, if nothing happens in the present, then nothing happens at all, and the question is: how does it get into the past—then, how can it be the case that it had happened? Did it ever happen ? No. Then how can it have happened?

(4) Another difficulty is presented by the phenomenon of motion or, in a more general way, by the phenomenon of functional relationship between two variables. Taking the relatively more concrete case of motion, let us assume that we are in a room whose doors have been barred. Now, let a ball move from near one of the walls, W1, towards the opposite wall, W2. Let us assume that before the ball went from W2 to W2, it had remained at rest for a certain period of time, t1, and that the ball took a period of time 12 in crossing the room and reaching near W2. Let us assume that the ball's motion was continuous. During time 13, the ball is once again at rest at the spot it reached near W. Let us further assume that t1, t2 and 13 are parts of time t such that tt+12+t3=t. Now, during the interval of time it is in motion, i.e. during t2, the ball was nowhere outside the room; during t1 and 13, before and after its fight (?), it is indubitably (since, admittedly) somewhere in the room, and hence, it is nowhere outside the room. During the whole of time t, the ball is nowhere outside the room. If we subtract the room from the universe, we can say that the ball in question does not exist during the period of time 12. The ball is indubitably in the room during t1 and 13 ; the room remains completely closed during the whole of the period t; in no sense (i.e. on no hypothesis, finposatomic, posinfible, infinzeratomic, or any other [except a theory of re-creation]) is the ball anywhere outside the room during any part of time t ; the ball is throughout in the room during t on the finposatomic, the posinfible and the infinzeratomic hypotheses, i.e. there is a

sense in which the ball remains in the room during t2; therefore, let us assume

that the ball does remain in the room during 2, that is to say, during the whole of time t. Assuming that the motion of the ball, in going from W1 to W2, is continuous and that its speed remains the same throughout the period in question, we know that if ta be the first half of 2 and tb the second half, then the ball will have traversed the first half of W1-W2 in ta and the second half of W1-W2 in tb. We may continue the process and learn about shorter and shorter periods of time as to which (shorter and shorter) distance was traversed by the ball during that shorter period; but we shall not to able to determine the place where the ball is during any given sub-period of t2. Some philosophers have, therefore, concluded that the ball is nowhere during the period of its motion, it is ever engaged in passing from place to place. This view seems to be no less paradoxical. If during the period of its motion, the ball is supposed to be just nowhere, then two questions appear to suggest them-selves One, what happens to the ball while it is supposedly in motion? Does it continue to exist? If so, where—inside, or outside the room? Two, is there any essential difference between saying that the ball, while in motion, is nowhere inside the room and saying that it is nowhere outside the room? It seems that neither an affirmative nor a negative answer is at all possible, and hence that the assumption of imposinfibility is incompatible with the phenomenon of motion.

To begin with, we are unable to accept that the ball goes out of existence altogether when in a state of motion. What is sup-posed to be in a state of motion, if not the ball? And if the ball becomes non-existent, how is it that it, neverthelese, is in a state of motion? Are "to be" and "to be in a state of motion" contradictories of each other? The ball, we assume, continues to exist, and, therefore, it must be somewhere. Since the room is closed and it cannot get out of it, it must throughout have been within the room. But where exactly, granted that it is in the room? And this is the crux of the problem: determining the position of the mobile at a given moment, just as it can be done if space and time are supposed to be finposatomic in composition.

Let us take up the other question. It seems to be obvious that there must be some fundamental difference between the two statements. For, obviously, it cannot be in the same sense of "being nowhere" that the ball is nowhere in and is nowhere outside the room. When at rest, the ball is in a specifiable space within the room and is nowhere outside the room in such a sense that if the ball were not in the room it would simply not exist. But now it is nowhere in the room and yet it (supposedly) continues to exist, even though it is nowhere outside the room in the same sense in which it was nowhere outside the room. In other words, the ball must be in the room even though it is, at the same time, nowhere in the room, while it neither is outside the room (in a general way) nor anywhere outside the room. To make the difference more striking, while in the room it is possible to let the ball strike our hands, it is not possible to let it do so outside the room (since outside the room the given ball, by assumption, simply does not exist), and, yet, it is nowhere either in the one or the other (i.e. the space inside or the space outside the room).[121]

In short, there must be, as said earlier, some fundamental difference between the two, but, within the chosen system, we are unable to see how to state this difference. Hence, the hypothesis of imposinfibility must he given up.

(5) Yet another difficulty is involved/implicit in imposinfibility, which we may refer to as the "Which-First?" dilemma to make the nature of the difficulty clearer (and later its solution easier). We had assumed in the preceding paragraph that the ball had a speed of zero space units per time unit during the whole of t1 (and again during t3), while throughout the whole of 2 it had a certain positive speed, x units of space per time unit. We had taken no note of the oddity involved in such an assumption. While we had demanded continuity of spatial existence, we had overlooked the fact that the ball's speed had jumped from o to x without having to go all the way from o to x. If the ball is not allowed by us to go from place A to place B without traversing the distance between A and B, how can we allow tile ball to reach the speed of x without its speed having to traverse the distance between o

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[121] If, for the time being, we revert to the assumption of space and time being finposatomic, then the desired distinction can he effected quite satisfactorily : The arrow is nowhere outside the room in the sense that at no microchrone during the given interval of time does it occupy any space out-side the room ; and it is throughout in the room, in the sense that during each microchrone of the given period of time it occupies specifiable space inside the room. And specifiable space here means a specifiable microtope or a specifiable collection of microtopes.

and x speeds ? When a car is started, does its speed jump from 0 miles per hour to say, 30 m.p.h. acquiring the speeds of, to mention only a few, 5 m.p.h., 10 m.p.h. or 1 S m.p.h. ? After all, why has the infible hypothesis been adopted as against the non-infible (i.e. the atomistic) hypothesis? Obviously, to have the feeling of smooth transitions without jerks and jolts—in other words, be-cause of the restrictions on divisibility. Let us, therefore, go back to the room, and assume that 2 is divisible into three periods, t2a, t2b and t2, such that t2a+t2b + t2c = t2 and that while the ball moves with the uniform velocity of x in t2b, the ball so moves during t2a, and t2c that its speed rises continuously from 0 and goes to x during and falls from x to 0 during t2c. While we shall look for the place occupied by the ball during t2b,, we shall be looking for the position occupied by its speed during t2a or t2c. Now let us try to find out its getting into motion: during which sub-period of t2a does it occur, and what is the ball's speed during that period. Now, during any part of t2a the ball would be found to be in motion and hence to have a positive speed. As we approach the period t1 backwards (by dividing and sub-dividing 12a), we shall approach the speed of zero (the "speed" during 4), but no matter what part of t2a we select, the ball would have a positive speed. We will never reach the sub-period during which alone (and not during any of its proper parts) the ball is first in motion, nor the speed which it attains immediately on getting into motion. Thus, it would seem that imposinfibility is self-stultifying: it fails to provide the facilities in the hope of which it may be adopted—radical continuity. Moreover, if we do assume that just as a distance cannot be traversed without first traversing a part thereof, and that likewise it is not possible for the ball to move with the overall speed of y during t2 (or any sub-period of t2) without it being the case that there is a proper sub-period of t2 (or of the sub-period of t2 in question) during which the overall speed of the ball is, say, one half of y, then imposinfibility would seem to be impossible.

(6) Another difficulty inhering/inherent in imposinfibility might be presented as Zeno's Metrical Paradox of Extension.[122] But that would have

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122 The argument that if the infinitude of the parts of an infible unit are magnitudeless, then the given unit must itself be magnitudeless and, if the parts be magnitudinous, then the unit must be infinite in magnitude. This is how the Zenonian argument reported by Simplicius [H. Ritter and L. Preller, Historic Philosophiae Graecae (8th ed. Gotha, 1898)), fragments 133 and 135), has been construed. Another of Zeno's arguments, the one known as the

the effect of hiding the real difficulty and presenting another difficulty, the one which we have already discussed, viz. that x1+x2+ x3+ ... is not equal to x. The real difficulty may be presented through the example of a train that traverses the distance between Rawalpindi and Karachi but does not reach Karachi., and yet remains in Karachi before undertaking the return journey. The train in question is in Rawalpindi during a certain period t1, it traverses the distance between Rawalpindi and Karachi during period t2, remains in Karachi during period t3, and traverses the distance between Karachi and Rawalpindi during period t4, and is once again in Rawalpindi during period t5. It is given that period t=t1+t2+t3+t4+t5, and that t1 is followed by 2 which is followed by t3, t4 and t5 in that order. The difficulty is that the train does not reach Karachi, yet it manages to stay for the whole of a positive period, t3, in Karachi, without there being a period of time between 2 and t3 The difficulty becomes very acute if we assume that the run of the train is uninterrupted, i.e. the train does not stop at any of the inter. mediary/way stations. Suppose that the railway track between Rawalpindi and Karachi goes via Lahore, but that our train does not stop at Lahore. We may, if we so wish, say that the train traverses the distance between Rawalpindi and Lahore as also the distance between Lahore and Karachi, but we cannot say that the train ever was in Lahore—for there is no period of time tx such that it is a part of t3 and during which the train is in Lahore. Thus, though the train could be said in some sense to pass through Lahore, it cannot be said to have been to Lahore. (This also is perhaps not a very good way of presenting the difficulty. The difficulty may be exhibited more simply, and more directly if not any more dramatically—as the difficulty of having no magnitudeless points and moments in addition to the positive intervals, with the consequence that while a distance is traversed, no position can be reached as a result thereof.)

(7) Achilles grants a finite handicap to a tortoise and runs a race with him,, Suppose he takes t time to traverse x, the handicap distance. If the

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"Race-Course" or as the "Dichotomy" (which we designated as the "Fixed-Goal Argument" in some of our works) has been construed either as one part of the Zero-or-Infinite argument or as the clam that an infinite set of tasks cannot be performed. But we distinguish between the Metrical Paradox of Extension and the Race-Cou~se Argument, interpreting the latter as the claim that point p2 cannot come to be occupied from any point p1 as a result of even traversing all of an infinitude of sub-distances—as a result of traversing the distance between p1 and p2.

speed of tortoise is a/b that of Achilles' speed (a and b are any two natural numbers such that b> a), in t time the tortoise will have traversed (x) distance. Achilles will take a/b (t) time to traverse a/b (x) distance, during which time the tor-a2 wise will have traversed a2/b2 (x) distance. Since, space and time are by assumption infible, there will ever remain a distance for Achilles to traverse to be able to catch the tortoise. The fact that we know where and when Achilles will overtake the tortoise (or in what period of time and over which length of space will the race be run) does not seem to be relevant, for what is involved is the question whether the overtaking is possible on the imposinifible hypo-thesis. There are two difficulties here: one, completion of an infinite sequence of runs, and, two, there being no positions (points) on the hypothesis of imposinifibility. We have maintained that infibility does not entail infinity of parts,[123] but, now, it would seem that an infinity of parts is inescapable. Achilles-tortoise race can-not involve anything less than a w-sequence of runs. And if we assume that Achilles must overtake the tortoise, we must be pre-supposing that an infinite process can come to an end. This part of the argument is applicable generally to the infible hypothesis. But the second argument involves a special difficulty of the imposinible hypothesis. Even if we assume that an infinite sequence of runs have come to pass, we fail to see how the tortoise would be overtaken, since the "point of overtaking" simply does not exist in an imposinfible race-course.

(8) This shows that there could be no Calculus on the assumption of imposinfibility. The Calculus might not stand in need of the notion of an infinitesimal magnitude, it does stand in need of the notion of "instantaneous rate of change," and this is obviously an insignificant juxtaposition of words on the hypothesis of imposinfibility. There being no "instants" on this assumption, there can be no sense in the expression "instantaneous rate of change". It is possible that a body may traverse the distance din one hour, two d distances in the next hour, and five d during the third hour. The speed of the body is not the same throughout the period of its motion, But we cannot describe the situation as it is done with the help of the notion of a differential coefficient.

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[123] In "Zeno's Paradoxes: Towards a Solution At Last," op, cit., Vol. XI (1972), pp. 125-51 ; see pp. 13032.

When we look at the arguments advanced against the imposinfible hypothesis, we may feel that all these arise because of the absence of what we call "the point" (and appropriate temporal and other counterparts), and that the postulation of points in infible spatial interval (and of moments, in temporal intervals, etc.) would make these arguments pointless if directed against infible units having points, moments, etc. If only we would have end-parts, as we have them in the (?) finposatomic units, we may feel, all the shortcomings of the imposinfible units enumerated above would have been obviated: "being somewhere," far from having to become a contradictory of "being in motion," would become quite irrelevant to "being in motion" as well as to "being at rest"; the science of geometry as we know it would become possible (once again !?): "instantaneous rate" would acquire a meaning and hence the Calculus would acquire a logical base; the threat posed to modern physics would have been removed and the theories in question would have been rehabilitated ; and, above all, the impalpability and fluidity inhering in imposinfible units would have been removed by the postulation of end-parts, which would have made any given unit, and any part thereof, as determinate as any finposatomic unit can be.

But no sooner shall we have desired to postulate such things as "points" than a question would suggest itself: how can we postulate an indivisible part of an infible unit? Is there no contradiction involved in the statements: "For any value of x, if x is a part of space/time, then x is divisible," and, "There is an x such that x is a part of space/time and x is indivisible"? The self-contradiction involved is too obvious to escape notice. Thus, it would seem that while it is in itself possible to postulate indivisible parts (= finposatomic or infinzeratornie hypothesis) the infible hypothesisis incompatible with the assumption of there being indivisible parts. This difficulty may be sought to be overcome by recalling that whole-part relationship is not invariably of the same kind, that it is not necessary that the part should be of the same logical type as the whole, and hence postulating that while it is necessary for a "component" of an infible unit to be itself divisible, it is not necessary for a "constituent" of an infible unit to do likewise. As no (finite) collection of such constituents could give rise to a unit of positive magnitude, we would be obliged to assume an infinity of constituents for any unit of positive magnitude--in short, we would he led to the infinzeratomio hypothesis. But we have elsewhere shown that the infinzeratomic hypothesis is self-contradictory.[124] Hence, we are obliged to give up the assumption that there can be indivisible parts—of any type whatsoever, components, constituents, or what you will—of an infible unit.

If, however, we are not allowed to postulate indivisible parts, then how can we postulate there being such things as "points"? Can we conceive of something which is assumed to have the twin virtue of not being a part of any unit of positive magnitude, and of having no magnitude? In other words, if a "point" be some-thing which has no magnitude and which is not a component of any positive interval of space, can we conceive of something that could answer to this description? If we were to divide and sub-divide a given unit in the hope of reaching such a degenerate interval as a point or moment, we shell have engaged ourselves in an impossible task. The magnitudeless something cannot, exhypothesi, be arrived at by any such process. If not, then how can it be arrived at all? If we cannot arrive at it from something having positive magnitude, then it cannot be supposed to subsist in anything having positive magnitude. Having no magnitude, it is plain nothing and cannot subsist by itself. In short, it is simply inconceivable.[125]

Notwithstanding the foregoing argument, even if we were to postulate something magnitudeless, like a "point" or a "moment," the question would be: can such a plain nothing be in any way related to things having positive magnitudes—could, for example, a (magnitudeless) point occur on or be contained in a line (of positive magnitude) even though it is not a component thereof nor is it possessed of any magnitude, or a magnitudeless moment occur or be contained in a period of time even though it is not a component of any temporal interval nor does it have any magnitude? If we do postulate magnitudeless something, points or moments for example, as separating two parts of the given unit, then, apart from the fact that no separation takes place and as such the parts fail to arise, a question arises with regard to the inclusion of that magnitudeless something in the parts separated by it. If be a spatial interval of positive magnitude, and if we postulate a point C such that it helps divide x into parts, y and z, then the question is as to where is point

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124 "Infinzeratomicity," op. cit., Vol. XIII, No. 4 (December 1975), pp. 34-72.
125 Cf. Aristotle, Physics. 263-b 9-26.

C to be found—in y or in z, or in neither, or, per impossible, in both? In a sense C lies neither my nor in z, in the sense in which it does not lie in x ; but we have assumed that it does lie in x, and, hence, it must lie in either y or: if not in both—unless, of course, it lies in x but is continued neither in y nor in z. There does not appear to be any a priori ground why C should lie in y and not in z, or lie in z but not in y. Therefore we cannot assume that C lies in y, nor assume that it lies in z. We cannot obviously assume that C lies in both y and z. How can two mutually exclusive sub-sets have a common member? And, how can one thing become two things (in order to get into two mutually exclusive sub-sets)? The obvious answer to both the questions is, it is simply inconceivable. It would, there-fore, seem that C must be conceived as lying between y and z and not as lying in y or in z. If we "cut" the set of numbers x (0<x <- 1) into two mutually exclusive sub-sets, the set of numbers y such that 0 <y < 2 and the set of numbers z such that I/2 <z

then the number "1/2" will answer to our magnitudeless something —it is a value for x, but it is neither a value for y nor for z, and hence lies between the set of values for y and the set of values for z. In other words, it seems that we would be led into the acceptance of the infinzeratomic and not the posinfible hypothesis, Even if the purely logical problem of a point or moment, etc., lying in two mutually exclusive parts of spatio-temporal units, etc., is supposed capable of a satisfactory solution, there would arise a problem in relation to the phenomenon of being in different states during different periods of time. If period of time t=ti+t2, and if moment in be the end-point of t1 and the first moment of t2, then, given that a body A is white during t1 and green during 12, it would seem that none of a logically exhaustive set of alternatives with regard to being or not being white or green at m can be adopted, and, hence, that m cannot be postulated, at the very least, to be contained in both t1 and 2.40 At in, or, for that matter, at any moment or during any time, A is white but not green, or A is neither white nor green (which is here equivalent to being either of colour x such that x is other than white and green or being colourless), or, per impossible, A is both white and green. But, as a matter of logic, A cannot both be white and not be white (Law of Contradiction), and hence A cannot be both white and green ; it cannot be the case that A is neither white nor green, for A is, as a matter of logic, either of some colour or it is of no colour (Law of Excluded Middle), and it is assumed that everything must be of some colour, whence it would follow that if A is neither white not green then it must be of colour x (such that x is not the same white or green), but there is no reason why A must be of colour x at m when at no moment during t1 or 2 it is of any colour other than that of white and green, and, moreover, we can legitimately assume that during t body A is either white or green, and if during the whole of t it is white or green, how can it be of colour x at any moment contained in t ? ; it cannot be green at m, for during the whole of t body A is white and m is contained in ti ; and, finally, it cannot be white at m since A is green throughout the period 2 which contains moment m. Thus it is seen that body A can be in no relevant state of affairs, whence it follows that m cannot be contained in both t1 and t2 even if it is not that m cannot be contained in t altogether.

The phenomenon of change would seem to present yet another problem on the supposition that two contiguous parts of a unit share a point, moment, or the like. Given that period t=ti+t2. that body A is white during t1 and green during t2, and that moment in is shared by t1 and 2, it would again seem that none of an exhaustive set of alternatives can be adopted with regard to the question of A ceasing to be white and becoming green. When does A cease to be white and become green?. A cannot be supposed to cease to be white at any moment during t1, for, ex hypothesi, A is white during the whole of ti, and, as such, at any moment during t1 A must be supposed to be white ; A cannot be supposed to cease to be white at m since m is in t1 and at any moment during t1 A is white ; and A cannot be supposed to cease to be white at any moment during 2, for, ex hypothesi, A is green during the whole of 2, and, as such, at any moment during t2 it must be supposed to be already green. If so, then when does A cease to be white and become green? We may put it this way. There must be a moment in such that A ceases to be white at mc and becomes green; but, there is no moment in t1 such that it can be identical with mc, nor is there any moment in t2 such that it could be identical with mc, nor can m be identical with mc. If the answer is that there is no moment at which A ceases to be white, that it just is white during t1 and green during 2, then, apart from the charge of imposinfibility, we would be obliged to hold that at no moment does a thing go out of existence or come into existence. A thing, A, let us assume, does not exist during time t1, is in existence during t2, and does not exist during time t3, and that t1, t2 and t3 are mutually exclusive and t1±t2-}-t3—t such that t1 is followed by t2 which is succeeded by t3. Now, the question is: when does it come into existence, and when does it go out of existence ? Did it come into existence during t1, or during i2, or at moment m1 such that m1 is the junction between t1 and 2? Did A cease to exist during t2, or during t3 or at m2 such that m2 is the moment that lies in both t2 and t3? But A cannot come into existence during ti, for throughout the period t1 it does not exist ; it cannot come into existence during 2, for it is in existence throughout the period t2; and it cannot come into existence at m1 since m1 is contained in t1. and we have already held that A cannot come into existence during t1. And, similarly, A cannot cease to exist during t2 or during t3 or at m2. We might, therefore, be tempted to answer with Aristotle41 (as also with "modern" mathematitians) that it came into existence at moment ma which is the first moment of 2 (but which is not included in ti) and that it ceased to exist at moment mb which is the first moment of t3 (but which is not included in t2). But then we shall have given up the posinfible hypothesis, for, now, there would be no moment linking (and hence lying in both of) two non-overlapping periods of time, and shall have adopted something like the infinzeratomic hypothesis. If we do not adopt the Aristotelian hypothesis, then, it would seem that either the same moment be the last moment of being ungenerated and the first moment of existence, and the same moment be the last moment of life and the first moment of ceasing to be alive, or there must be two pairs of two separate moments such that one pair of moments consists of two moments of which one is the last moment of the state of being ungenerated and the other is the first moment of existence, and the other pair consists of two moments such that one is the last moment of being in existence/life and the other moment is the first moment of ceasing to be in existence/alive. But neither offers a tenable alter-native. At no moment can anything both be and not be; hence, there can be no moment at which a thing is both ungenerated and in existence. Again, there can be no two such moments, since there are no two consecutive moments on the infible hypothesis.[126]

[126] Ibid., 263-b 9 if. ("It is also evident that, when, speaking of the subject of motion or change, unless we assign the instant that divides past and future time to the state into which that object turns and in which it will be for the future rather than to that which it turns out of and in which it was in the past, we shall have to say that the same thing both exists and does not exist at the same instant, and when it has become something it is not that something it has become.")

We may present the argument differently. There is a phenomenon which is incompatible with the hypothesis of posinfibility, or, at least, a phenomenon which cannot be treated posinfibly. (Is the second alternative necessary? Can we not dispense with it?) If x and y be any two distinct (i.e. nonidentical) positive states[127] of body A and if x and y be consecutive, then there can be no S such that S is a degenerate state[128] and S is contained in both x and y. In other words, if period of time t=t1+t2, and if body A is in state x during t1 and in state y during t2, then if S be the end-point of x, then not only that S cannot be an end-point (first degenerate state) of y, y can have no first degenerate state at all, and if S be the first degenerate state of y, then not only that S cannot be the end-point of x, x can have no endpoint at all ; and this would seem to entail that if m be the end-point of t1, then not only that m cannot be an end-point (first moment) of t2, t, can have no first moment at all, and if m be the first moment of r2, then not only that m cannot be the end-point of t1, t1 can have no end-point at all. In short, there can be no degenerate state which lies in both of two consecutive positive states, and hence there can be no moment which lies in both of two consecutive periods of time. But, there can be (and, 'as a matter of fact, there are) two positive states of a given body such that the two are consecutive but are different from each other. Thus, all possible states of affairs cannot be dealt with on the posinfibie hypothesis; and if some one hypothesis has to account for everything, then we can-not adopt the posinfible hypothesis.

MY SWORD BELONGS TO MY SUPREME MASTER*

The First Great War culminated in the victory for the Allied arms. Turkey had joined her lot with Germany. Germany was defeated and Turkey sailed in the same boat.

An armistice was signed and Turkey was to surrender, among other territories, Medina to the English forces.

Fakhr-ud-Din Pasha, the Military Governor of Medina, declined to surrender the Sacred City to the foreign forces. The Sultan was informed and

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127 . A "positive state" is a state that persists for a positive interval of time. (Here "positive" is opposed to "degenerate" and not to "negative".)

128 A state may be said to be degenerate if it is momentary, i.e. if it does not last for any period of time, however short. in other words, a degenerate state is an end-point of a positive state.

he sent a seccond command for immediate surrender, but Fakhr-ud.Din was obdurate.

There was a long and cruel siege. The stock of food and water ran out and the suffering of the inhabitants knew no bounds. But Fakhr-ud-Din was still unmoved in his resolve: he would not surrender his sword to the foe.

At last his staff begged him to spare the lives of all of them what was certain starvation. Just at this moment a third order from the Sublime Porte arrived---an order for the immediate evacuation of Medina according to the terms of the armistice.

The heart of Fakhr-ud-Din broke. Deeply agonised he silently wended to the tomb of the Prophet and sobbed out: "My sword belongs to my Supreme Master; if I am to give it up for the sake of human life, it would be to him alone." With that he laid his sword at the foot of the holy sepulchre and swooned,

*From Halide Edib, The Turkish Ordeal